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Note

Elimination of Spurious Eigenvalues in the Chebyshev Tau Spectral Method

1. Introduction

Spectral methods have been used to great advantage in hydrodynamic stability calculations; the concepts are described in Orszag's seminal application of the Chebyshev tau method to the Orr–Sommerfeld equation for plane Poiseuille flow in 1971 [1]. Orszag discusses both the Chebyshev Galerkin and the Chebyshev tau methods, but presents results for the tau method, which is easier to implement than the Galerkin method. The tau method has the disadvantage that two unstable eigenvalues are produced that are artifacts of the discretization. The occurrence of spurious eigenvalues has been discussed by several authors, cf. [2–7].

In this note we present an extremely simple modification to the Chebyshev tau method which eliminates the spurious eigenvalues. We first study a simplified model of the Orr–Sommerfeld equation discussed by Gottlieb and Orszag [2]. We consider the Chebyshev tau method, which has two spurious eigenvalues and then describe a modification which eliminates them. Our modification is motivated by considering two other discretizations of the model problem which also have no spurious modes: a vorticity-streamfunction reformulation of the Chebyshev tau method, and the Chebyshev Galerkin method. For the model problem we show that the latter approaches are equivalent and that both reduce to our modification of the tau method. We also remark on the modified Galerkin method formulated by Zebib [4, 6]. Finally, we consider results for the Orr–Sommerfeld equation, where our modified tau method also eliminates the spurious eigenvalues. The simplicity of the modification makes it a convenient alternative to other approaches to the problem.

2. The Model Problem

In their 1977 monograph on spectral methods, Gottlieb and Orszag [2, p. 143] consider the problem

$$\Psi_{zzt} = \Psi_{zzzz} \qquad (-1 < z < 1,\ t > 0), \tag{1}$$

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with the boundary conditions $\Psi(\pm 1, t) = \Psi_z(\pm 1, t) = 0$, as a simple model of incompressible fluid dynamics. A normal mode of the form $\Psi(z, t) = \psi(z) \exp \sigma t$ gives rise to the eigenvalue problem

$$\psi_{zzzz} = \sigma \psi_{zz} \qquad (-1 < z < 1), \tag{2a}$$

with the boundary conditions

$$\psi(\pm 1) = \psi_z(\pm 1) = 0. \tag{2b}$$

The latter problem models some features of the Orr–Sommerfeld equation [1] and can be solved exactly. The eigenvalues of (2) are given by the values $\sigma = -\mu^2$, where either $\mu = n\pi$ for $n = 1, 2, \ldots$, corresponding to the even eigenfunctions with $\psi_{zz} = \cos \mu z$, or μ is a positive root of the transcendental equation $\tan \mu = \mu$, corresponding to the odd eigenfunctions with $\psi_{zz} = \sin \mu z$.

2.1. *The Tau Method*

Gottlieb and Orszag [2] show that a straightforward application of the Chebyshev tau method to Eq. (2) gives rise to positive eigenvalues whose magnitudes increase rapidly as more terms are included in the expansion. Such spurious modes may be discarded by inspection for the system given by (2), but they cause severe numerical instability in the time-dependent system (1). The equations for the Chebyshev tau method are obtained by writing

$$\psi = \sum_{n=0}^{N} a_n T_n(z), \tag{3}$$

where $T_n(z)$ is the nth degree Chebyshev polynomial, which satisfies $T_n(\cos \theta) = \cos n\theta$. We then have [1]

$$\psi_{zz} = \sum_{n=0}^{N} a_n^{(2)} T_n(z), \qquad \psi_{zzzz} = \sum_{n=0}^{N} a_n^{(4)} T_n(z), \tag{4a}$$

where

$$c_n a_n^{(2)} = \sum_{\substack{p = n+2 \\ p+n \text{ even}}}^{N} p(p^2 - n^2)\, a_p, \tag{4b}$$

$$c_n a_n^{(4)} = \frac{1}{24} \sum_{\substack{p = n+4 \\ p+n \text{ even}}}^{N} p[p^2(p^2-4)^2 - 3n^2p^4 + 3n^4p^2 - n^2(n^2-4)^2]\, a_p, \tag{4c}$$

and $c_0 = 2$ and $c_n = 1$ for $n > 0$. Note that these expressions give $0 = a_{N-1}^{(2)} = a_N^{(2)} = a_{N-3}^{(4)} = a_{N-2}^{(4)} = a_{N-1}^{(4)} = a_N^{(4)}$, since ψ_{zz} and ψ_{zzzz} are polynomials of degree $N-2$ and $N-4$, respectively.

Equations for the coefficients a_n are defined by the relations

$$\left(T_j, \frac{d^4\psi}{dz^4}\right) = \sigma\left(T_j, \frac{d^2\psi}{dz^2}\right), \qquad j = 0, ..., N-4,$$

where in the Chebyshev inner product we have $(T_j, T_k) = (\pi/2)\, c_j \delta_{jk}$. The resulting tau equations are then

$$a_n^{(4)} = \sigma a_n^{(2)}, \qquad n = 0, ..., N-4, \tag{5a}$$

with the four boundary conditions that follow from (2b),

$$\sum_{n=0}^{N} (\pm 1)^n\, a_n = \sum_{n=0}^{N} (\pm 1)^n\, n^2 a_n = 0. \tag{5b}$$

The equations can be written as a generalized eigenvalue problem $Ax = \sigma Bx$, where A and B are $(N+1) \times (N+1)$ matrices whose first $N-3$ rows are defined by Eq. (5a). The last four rows of A are given by Eq. (5b), and the last four rows of B vanish.

We illustrate the matrix equations schematically for the case $N = 8$:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & x & \cdot & x & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \sigma \begin{bmatrix} \cdot & \cdot & x & \cdot & x & \cdot & x & \cdot & x \\ \cdot & \cdot & \cdot & x & \cdot & x & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & \cdot & x & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}; \tag{6}$$

here the non-zero entries are denoted by the symbol "x."

This system may be solved numerically using software such as the routine RGG from the EIS PACK library [8]. In Table I we give numerical results for the system (5) obtained with RGG. The table shows the values of the first and fifth negative eigenvalues σ_1 and σ_5, and the larger of two positive spurious eigenvalues $\sigma_{\max}$. These results reproduce those given in Table 13.1 of [2]. (Note that there is a typographical error in [2] in the results for σ_5 for large N.) The numbers were computed in double precision on the CDC Cyber 205 at the National Institute of Standards and Technology to reduce the round-off error. For single precision calculations on the Cyber 205 the effect of round-off becomes noticeable in the last two digits of σ_1 for $N > 25$; the given values for σ_5 and $\sigma_{\max}$ remain the same.

Subroutine RGG provides solutions to the problem written in the form $\beta Ax = \alpha Bx$ [9], so that for $\beta_j \neq 0$ the jth eigenvalue is given by $\sigma_j = \alpha_j/\beta_j$. Solutions

TABLE I

Eigenvalues for the Model Problem

	Chebyshev tau [2]			Present results	
N	σ_1	σ_5	$\sigma_{\max}$	σ_1	σ_5
10	-9.8696598	-189.63800	4,272.	-9.8695970	-97.95740
15	-9.8696044	-89.54550	29,439.	-9.8696044	-88.84327
20	-9.8696044	-88.82644	111,226.	-9.8696044	-88.82644
25	-9.8696044	-88.82644	294,697.	-9.8696044	-88.82644
30	-9.8696044	-88.82644	652,722.	-9.8696044	-88.82644
35	-9.8696044	-88.82644	1,255,298.	-9.8696044	-88.82644
Exact	-9.8696044	-88.82644		-9.8696044	-88.82644

with $\beta = 0$ correspond to the null space for the related problem $Bx = \mu Ax$, where $\mu = 1/\sigma$, and may be associated with infinite values for σ. Eigenvalues with $\beta = 0$ occur for the above system due to the boundary conditions (5b). For a given value of $N > 6$, four eigenvectors with $\beta = 0$ are obtained, and the other $N-3$ eigenvectors have $\beta \neq 0$. Eigenvalues with $\beta = 0$ may be avoided by using the four equations (5b) representing the boundary conditions to eliminate the variables a_{N-3}, a_{N-2}, a_{N-1}, and a_N from the system as described in [7]. This eliminates the eigenvalues with $\beta = 0$; the values with $\beta \neq 0$ are unchanged. The numbers α_j and β_j can also be used [10] to give an estimate of the sensitivity of the computed eigenvalue to perturbations in A and B: σ_j is ill-conditioned if the condition number $C(\sigma_j) = 1/\sqrt{|\alpha_j|^2 + |\beta_j|^2}$ is large, in which case σ_j will be difficult to determine using low precision arithmetic.

2.2. *A Modified Tau Method*

A useful modification of the tau method (5) is obtained by considering instead the equations

$$a_n^{(4)} = \sigma \bar{a}_n^{(2)}, \qquad n = 0, ..., N-4, \tag{7a}$$

with the four boundary conditions

$$\sum_{n=0}^{N} (\pm 1)^n a_n = \sum_{n=0}^{N} (\pm 1)^n n^2 a_n = 0, \tag{7b}$$

where we have defined

$$c_n \bar{a}_n^{(2)} = \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{N-2} p(p^2 - n^2)\, a_p. \tag{8}$$

Note that $\bar{a}_n^{(2)}$ is obtained from $a_n^{(2)}$ by simply dropping the last two terms. Thus, the modified matrix eigenvalue problem is obtained from the tau eigenvalue problem $Ax = \sigma Bx$ by setting the last two columns of B to zero.

Numerical results for the system (7) are also presented in Table I. The two spurious eigenvalues that are obtained using the standard tau method are eliminated, and the remaining eigenvalues are computed with essentially no loss in accuracy. The numbers were computed in double precision on the CDC Cyber 205 at NIST.

The motivation for our modification of the usual tau method is given in the following two sections.

2.3. *Vorticity-Streamfunction Formulation*

Gottlieb and Orszag [2] show that the spurious roots in the Chebyshev tau method are eliminated if the function $\zeta = \psi_{zz}$ is introduced; ζ plays the role of vorticity in this simple model. In this case the model equations take the form

$$\zeta_{zz} = \sigma\zeta, \tag{9a}$$

$$\psi_{zz} = \zeta, \tag{9b}$$

with

$$\psi(\pm 1) = \psi_z(\pm 1) = 0. \tag{9c}$$

We now have two coupled second-order equations for ζ and ψ; note, however, that there are four boundary conditions on ψ and none on ζ.

If, in addition to Eq. (3), we write

$$\zeta = \sum_{n=0}^{N} b_n T_n(z), \tag{10}$$

then the tau equations for (9) become

$$b_n^{(2)} = \sigma b_n, \qquad n = 0, ..., N-2, \tag{11a}$$

$$a_n^{(2)} = b_n, \qquad n = 0, ..., N-2, \tag{11b}$$

together with the four boundary conditions

$$\sum_{n=0}^{N} (\pm 1)^n a_n = \sum_{n=0}^{N} (\pm 1)^n n^2 a_n = 0. \tag{11c}$$

These equations also take the form of a generalized eigenvalue problem and may be solved using subroutine RGG. In this case it is found that the discretized Poisson equation (11b) contributes $N+1$ eigenvectors with $\beta = 0$. There are an additional four eigenvectors with $\beta = 0$ arising from the terms representing the boundary conditions, leaving $N-3$ eigenvectors with $\beta \neq 0$. For this system all eigenvalues with $\beta \neq 0$ are negative; the spurious eigenvalues have been eliminated.

As pointed out by Gardner et al. [7], the size of the system may be reduced by using the last two equations in (11a) to eliminate the unknowns b_{N-1} and b_N, and then using (11b) to eliminate b_j for $j=0, ..., N-2$. The first $N-3$ equations in (11a), together with the boundary conditions (11c), then constitute a generalized eigenvalue problem analogous to the system (5), but without spurious eigenvalues that have $\mathrm{Re}(\sigma)>0$. The eigenvalues with $\beta \neq 0$ are unchanged by the elimination. We next show explicitly the difference between formulations (5) and (11). If we introduce the general notation

$$b_j^{(2)} = \sum_{k=0}^{N} \Gamma_{jk} b_k, \qquad j=0, ..., N,$$

where the elements Γ_{jk} follow from the explicit form of Eq. (4b), then the last two equations in (11a) give

$$\sigma b_{N-2} = b_{N-2}^{(2)} = \Gamma_{N-2,N} b_N, \qquad \sigma b_{N-3} = b_{N-3}^{(2)} = \Gamma_{N-3,N-1} b_{N-1}.$$

Putting these expressions for b_N and b_{N-1} into the first $N-3$ equations in (11a), and then using (11b) to eliminate the remaining b_j, we obtain

$$\sum_{k=0}^{N-2} \Gamma_{jk} a_k^{(2)} = \sigma \left(a_j^{(2)} - \frac{\Gamma_{j,N-1}}{\Gamma_{N-3,N-1}} a_{N-3}^{(2)} - \frac{\Gamma_{j,N}}{\Gamma_{N-2,N}} a_{N-2}^{(2)} \right), \qquad j=0, ..., N-4.$$

Since $a_{N-1}^{(2)} = a_N^{(2)} = 0$, the left-hand side may be written

$$\sum_{k=0}^{N-2} \Gamma_{jk} a_k^{(2)} = \sum_{k=0}^{N} \Gamma_{jk} a_k^{(2)} \equiv a_j^{(4)}, \qquad j=0, ..., N.$$

Finally, since $a_{N-3}^{(2)} = \Gamma_{N-3,N-1} a_{N-1}$ and $a_{N-2}^{(2)} = \Gamma_{N-2,N} a_N$, we conclude that the tau equations (11a) and (11b) are equivalent to

$$a_n^{(4)} = \sigma a_n^{(2)} - \sigma \Gamma_{n,N-1} a_{N-1} - \sigma \Gamma_{n,N} a_N, \qquad n=0, ..., N-4. \tag{12}$$

Since

$$a_n^{(2)} = \sum_{k=0}^{N} \Gamma_{nk} a_k, \qquad n=0, ..., N,$$

the resulting system is precisely equivalent to our modified equations (7).

2.4. *Chebyshev Galerkin Method*

The Chebyshev Galerkin discretization of the model problem (2) is also free from spurious eigenvalues [2]. In this representation, we write

$$\psi = \sum_{n=4}^{N} a_n \phi_n(z), \tag{13}$$

where the functions ϕ_n are chosen to satisfy the boundary conditions $\phi_n = d\phi_n/dz = 0$ at $z = \pm 1$, i.e.,

$$\phi_n(z) = T_n(z) + \gamma_{n0} T_0(z) + \gamma_{n1} T_1(z) + \gamma_{n2} T_2(z) + \gamma_{n3} T_3(z), \qquad n = 4, ..., N;$$

here

$$\gamma_{n0} = \begin{cases} \frac{1}{4}(n^2 - 4), & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \qquad \gamma_{n1} = \begin{cases} 0, & n \text{ even} \\ \frac{1}{8}(n^2 - 9), & n \text{ odd} \end{cases}$$

$$\gamma_{n2} = \begin{cases} -\frac{1}{4}n^2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \qquad \gamma_{n3} = \begin{cases} 0, & n \text{ even} \\ -\frac{1}{8}(n^2 - 1), & n \text{ odd.} \end{cases}$$

The Galerkin equations for the coefficients a_j are defined by the relations

$$\left(\phi_j, \frac{d^4\psi}{dz^4}\right) = \sigma\left(\phi_j, \frac{d^2\psi}{dz^2}\right), \qquad j = 4, ..., N. \tag{14}$$

If we introduce four additional coefficients a_0, a_1, a_2, and a_3, we may write as before

$$\psi(z) = \sum_{n=0}^{N} a_n T_n(z), \tag{15}$$

where

$$a_j = \sum_{n=4}^{N} a_n \gamma_{nj}, \qquad j = 0, ..., 3.$$

It follows that

$$\frac{d^4\psi}{dz^4} - \sigma \frac{d^2\psi}{dz^2} = \sum_{n=0}^{N} e_n T_n(z), \tag{16}$$

where $e_n = a_n^{(4)} - \sigma a_n^{(2)}$. The Galerkin equations (14) are therefore equivalent to the $N-3$ equations,

$$0 = e_j + 2\gamma_{j0} e_0 + \gamma_{j1} e_1 + \gamma_{j2} e_2 + \gamma_{j3} e_3, \qquad j = 4, ..., N, \tag{17a}$$

plus the boundary conditions

$$\sum_{n=0}^{N} (\pm 1)^n a_n = \sum_{n=0}^{N} (\pm 1)^n n^2 a_n = 0, \tag{17b}$$

which hold by the construction of a_0, a_1, a_2, and a_3.

We next show that Eqs. (17a) are equivalent to (12). Since $\psi(z)$ is a polynomial of degree N, it follows from (16) that $e_N = e_{N-1} = 0$, $e_{N-2} = -\sigma \Gamma_{N-2,N} a_N$, and

$e_{N-3} = -\sigma\Gamma_{N-3,N-1}a_{N-1}$. The last four equations in (17a) may then be used to obtain e_0, e_1, e_2, and e_3 in terms of a_N and a_{N-1}; a calculation gives

$$e_k = -\sigma\Gamma_{k,N-1}a_{N-1} - \sigma\Gamma_{k,N}a_N, \qquad k=0, ..., 3.$$

A further calculation then shows that

$$2\gamma_{j0}e_0 + \gamma_{j1}e_1 + \gamma_{j2}e_2 + \gamma_{j3}e_3 = \sigma\Gamma_{j,N-1}a_{N-1} + \sigma\Gamma_{j,N}a_N, \qquad j=4, ..., N-4.$$

Combining these last two expressions with (17a) we obtain

$$e_n = -\sigma\Gamma_{n,N-1}a_{N-1} - \sigma\Gamma_{n,N}a_N, \qquad n=0, ..., N-4,$$

which reduce to (12).

We conclude that each of the discrete eigenvalue problems (7), (11), and (17) are equivalent in the sense that eigenvalues σ with $\beta \neq 0$ that are produced by each formulation are identical.

2.5. *Other Galerkin Procedures*

Other variants of the Galerkin formulation are possible. For example, Zebib [4, 6] introduces a different basis by writing

$$\psi = \sum_{n=0}^{N-4} h_n u_n(z), \tag{18}$$

where the $u_n(z)$ are linearly independent polynomials of degree at most N, uniquely characterized by

$$\frac{d^4u_n}{dz^4} = T_n(z),$$

with the boundary conditions $u_n = du_n/dz = 0$ at $z = \pm 1$. Zebib's original approach [4] produced results that included spurious eigenvalues, which he was later able to eliminate [6].

The functions $u_n(z)$ for $n=0, ..., N-4$, and the functions ϕ_k for $k=4, ..., N$ used in the previous section span the same space, i.e., the set of Nth degree polynomials which, together with their derivatives, vanish at $z = \pm 1$. It follows that the function $\psi(z)$ in Eq. (18) can be re-expressed as a linear combination of $T_n(z)$, $n=0, ..., N$, as in Eq. (15), where the coefficients a_j are linearly related to the constants h_n above and satisfy (17b). Equation (16) holds as well.

Using Zebib's original approach [4] on the model problem, we take the inner product of Eqs. (16) with T_n for $n=0, ..., N-4$ and obtain equations which, in terms of the variables a_j, are precisely equivalent to the Chebyshev tau equations $e_n = 0$ for $n=0, ..., N-4$.

Using Zebib's later approach [6], we take the inner product of Eqs. (16) with u_n for $n=0, ..., N-4$. Since each basis function u_n can be expressed as a linear combination of the functions ϕ_k, we obtain a linear combination of Eqs. (17). The corresponding matrix equations are now related by pre- and post-multiplication by invertible matrices, and the spectra are identical. Thus Zebib's second approach produces eigenvalues which are in principle identical to those obtained using the regular Galerkin formulation outlined in the previous section. In practice, the condition numbers for the eigenvalues obtained using Zebib's later approach are large compared to those obtained using the standard Galerkin method. Roughly speaking, this may be attributed to the fact that the basis u_n is not as well-conditioned as the basis ϕ_k; the ϕ_k are more nearly orthogonal (cf. [11]).

These results generalize, in the sense that expanding in any basis for polynomials of degree N satisfying the boundary conditions still produces a function which satisfies Eqs. (15)–(16). Taking the inner product of the equations with T_n will produce the spectrum from the Chebyshev tau method, with spurious eigenvalues. Taking the inner product of the equations with the basis functions will produce the spectrum from the standard Galerkin method, with no spurious eigenvalues.

3. Other Model Problems

In Table II we list eight model problems of second through sixth order which were discretized and solved using the Chebyshev tau method with $N=20$. Spurious eigenvalues were observed in three of these cases. The first was considered in Section 2. The spurious modes in the remaining two problems, both sixth order, were eliminated using a modified tau method similar to that proposed in Section 2.2.

In problem (7) we use the modified discretization

$$a_n^{(6)} = \sigma \bar{\bar{a}}_n^{(2)}, \qquad n = 0, ..., N-6,$$

TABLE II

Occurrence of Spurious Eigenvalues in the Chebyshev Tau Method[a]

No.	Equation	Boundary Conditions	Spurious
1	$\psi^{(2)} = \sigma\psi$	$\psi(\pm 1) = 0$	None
2	$\psi^{(2)} = \sigma\psi$	$\psi^{(1)}(\pm 1) = 0$	None
3	$\psi^{(4)} = \sigma\psi$	$\psi(\pm 1) = \psi^{(1)}(\pm 1) = 0$	None
4	$\psi^{(4)} = \sigma\psi^{(2)}$	$\psi(\pm 1) = \psi^{(1)}(\pm 1) = 0$	Two
5	$\psi^{(4)} = \sigma\psi^{(2)}$	$\psi(\pm 1) = \psi^{(2)}(\pm 1) = 0$	None[b]
6	$\psi^{(6)} = \sigma\psi$	$\psi(\pm 1) = \psi^{(1)}(\pm 1) = \psi^{(2)}(\pm 1) = 0$	None
7	$\psi^{(6)} = \sigma\psi^{(2)}$	$\psi(\pm 1) = \psi^{(1)}(\pm 1) = \psi^{(2)}(\pm 1) = 0$	Two
8	$\psi^{(6)} = \sigma\psi^{(4)}$	$\psi(\pm 1) = \psi^{(1)}(\pm 1) = \psi^{(2)}(\pm 1) = 0$	Four[c]

a. Here we denote derivatives of ψ by a superscript in parentheses.

b. May be reduced to a Dirichlet problem for $\psi^{(2)}$.

c. Two complex conjugate pairs with $\mathrm{Re}(\sigma) < 0$, but $\mathrm{Im}(\sigma) \neq 0$.

where

$$c_n \bar{\bar{a}}_n^{(2)} = \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{N-4} p(p^2-n^2)\, a_p;$$

that is, the last four columns of the matrix representing $a_n^{(2)}$ are set to zero. In problem (8) we use the discretization

$$a_n^{(6)} = \sigma \bar{a}_n^{(4)}, \qquad n = 0, ..., N-6$$

where

$$c_n \bar{a}_n^{(4)} = \frac{1}{24} \sum_{\substack{p=n+4 \\ p+n \text{ even}}}^{N-2} p[p^2(p^2-4)^2 - 3n^2p^4 + 3n^4p^2 - n^2(n^2-4)^2]\, a_p;$$

that is, the last two columns in the matrix representing $a_n^{(4)}$ have been set to zero. In each of these cases no significant change in the accuracy of the computed eigenvalues was observed.

4. Orr–Sommerfeld Equation

The linear stability of a parallel viscous flow subject to spatially periodic disturbances is governed by the Orr–Sommerfeld equation, as described in standard texts on hydrodynamic stability (e.g., [12]). The Orr–Sommerfeld equation for plane Poiseuille flow may be written in the form

$$[\psi_{zzzz} - 2\alpha^2\psi_{zz} + \alpha^4\psi]/(i\alpha R) - (U(z)-s)(\psi_{zz} - \alpha^2\psi) + U_{zz}\psi = 0$$

for $-1 < z < 1$, where $U(z) = 1 - z^2$ is the base velocity, α is the wavenumber of the disturbance, R is the Reynolds number of the flow, and s is the temporal eigenvalue. The boundary conditions at $z = \pm 1$ are $\psi = \psi_z = 0$. For a given Reynolds number, the flow is stable if $\mathrm{Im}(s) < 0$ for all wavenumbers α. The critical Reynolds number R_c is that for which the imaginary part of s first vanishes at a critical wavenumber α_c as R is increased.

The use of the Chebyshev tau method to discretize the equations produces two spurious eigenvalues with $\mathrm{Im}(s) > 0$; results for the standard test case $\alpha = 1$ and $R = 10,000$ [1] are given in Table III. The results were obtained using single precision on the CDC Cyber 205 at NIST with the IMSL routine EIGZC [13].

The spurious eigenvalues may be removed by employing the vorticity-streamfunction formulation [2, 7] or by employing a Galerkin [2] or modified Galerkin approach [6]. As in the model problem, Zebib's approaches are expected to lead to the same eigenvalues as the Chebyshev tau [4] or Chebyshev Galerkin [6] methods (in the absence of rounding errors).

TABLE III

First Four Eigenvalues for the Orr–Sommerfeld Equation, $\alpha = 1$ and $R = 10^4$

N	Chebyshev tau	Present results
26	0.08079254 + 35.0190231 i	0.23627968 + 0.00445813 i
	0.08896405 + 29.3934954 i	0.89244759 − 0.02021587 i
	0.23713751 + 0.00563644 i	0.96105877 − 0.02035856 i
	0.76774613 − 0.00334424 i	0.90085138 − 0.02192943 i
38	0.05429957 + 178.047201 i	0.23752985 + 0.00373031 i
	0.05782029 + 158.755478 i	0.96336687 − 0.03521228 i
	0.23752676 + 0.00373427 i	0.96270591 − 0.03619040 i
	0.96383565 − 0.03503110 i	0.90798372 − 0.04519238 i
50	0.04092607 + 563.551154 i	0.23752648 + 0.00373967 i
	0.04288092 + 517.476254 i	0.96462731 − 0.03516958 i
	0.23752648 + 0.00373967 i	0.96464022 − 0.03518657 i
	0.96462865 − 0.03516827 i	0.27720546 − 0.05089517 i
From [1]	0.23752649 + 0.00373967 i	0.23752649 + 0.00373967 i
	0.96463092 − 0.03516728 i	0.96463092 − 0.03516728 i
	0.96464251 − 0.03518658 i	0.96464251 − 0.03518658 i
	0.27720434 − 0.05089873 i	0.27720434 − 0.05089873 i

As another alternative, a direct modification of the usual Chebyshev tau formulation is suggested by the results of Section 2.2. If the equation are written in the form

$$[\psi_{zzzz} - 2\alpha^2\psi_{zz} + \alpha^4\psi]/(-i\alpha R) + U(z)(\psi_{zz} - \alpha^2\psi) - U_{zz}\psi = s(\psi_{zz} - \alpha^2\psi),$$

then the Chebyshev tau formulation produces $N-3$ linear equations,

$$[a_n^{(4)} - 2\alpha^2 a_n^{(2)} + \alpha^4 a_n]/(-i\alpha R) + \cdots = s(a_n^{(2)} - \alpha^2 a_n), \qquad n = 0, ..., N-4, \quad (19)$$

together with the four boundary conditions (17b), where for simplicity we have omitted in (19) the convolution sums involving $U(z)$ whose form is unimportant for this discussion. To obtain the suggested modification, this expression is approximated instead by the formula

$$[a_n^{(4)} - 2\alpha^2 a_n^{(2)} + \alpha^4 a_n]/(-i\alpha R) + \cdots = s(\bar{a}_n^{(2)} - \alpha^2 a_n), \qquad n = 0, ..., N-4,$$

where the left-hand side is unchanged and $\bar{a}_n^{(2)}$ is given by (8); that is, the last two columns in the matrix that represents the second derivative are set to zero. As shown in Table III, this modification also serves to eliminate the two spurious eigenvalues in this case as well, with essentially no loss of accuracy in the other eigenvalues. It is not surprising that the modification successfully eliminates the spurious modes, since the Orr–Sommerfeld equation differs from the model problem only in the lower-order derivative terms on either side of the equation.

Due to differing treatment of these lower-order terms, this modification of the usual tau formulation of the Orr-Sommerfeld equation is not equivalent to either the vorticity-streamfunction formulation or the Galerkin formulation; the corresponding eigenvalues that are computed for a given N are similar but not identical. Our modified tau method is more efficient than the vorticity-streamfunction formulation [2], which doubles the size of the system, or a Galerkin formulation [1, 6], which is more awkward to implement. Eliminating half of the unknowns to reduce the size of the system in the vorticity-streamfunction formulation [7] also produces a more complicated system of equations.

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